Bending Stress in Beams

Derive a relationship for bending stress in a beam:

Basic Assumptions:

1. Deflections are very small with respect to the depth of the beam
2. Plane sections before bending remain plane after bending, ie. planes may rotate but not distort.
3. The material follows Hooke's Law - stress is proportional to strain $\sigma = E * \varepsilon$
4. Stresses are always in the linear range of the material and strains are fully recovered when the bending moment is removed.
5. Loads are applied slowly.

Definitions :

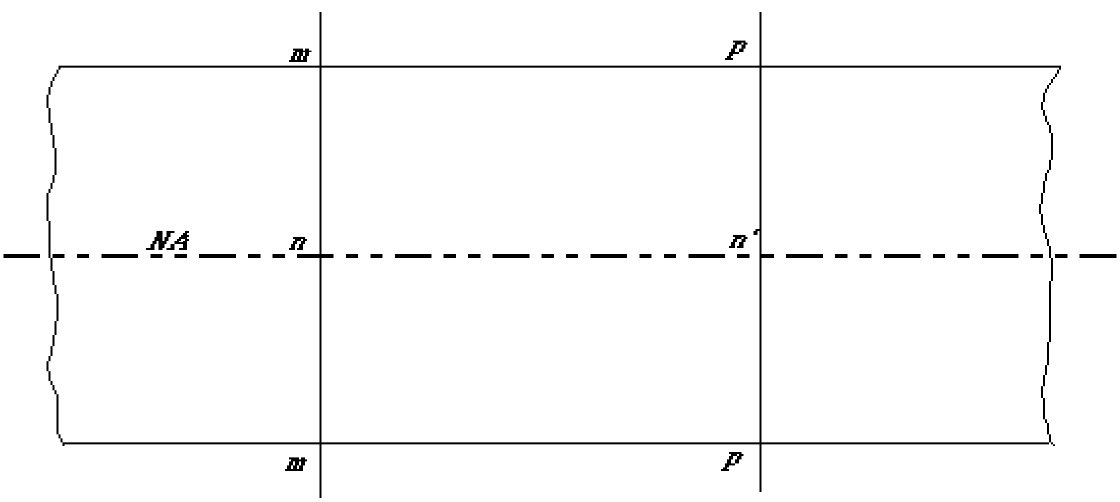
Strain is defined as change in length (ΔL) divided by the original length (L): $\varepsilon = \Delta L/L$

Stress is defined as a force(P) acting over an area (A): $\sigma = P/A$

Moment is defined as a force (P) acting through as distance (d): $M = P * d$

Moment of inertia $I = \int y^2 \, dA$ *(see any calculus text)*

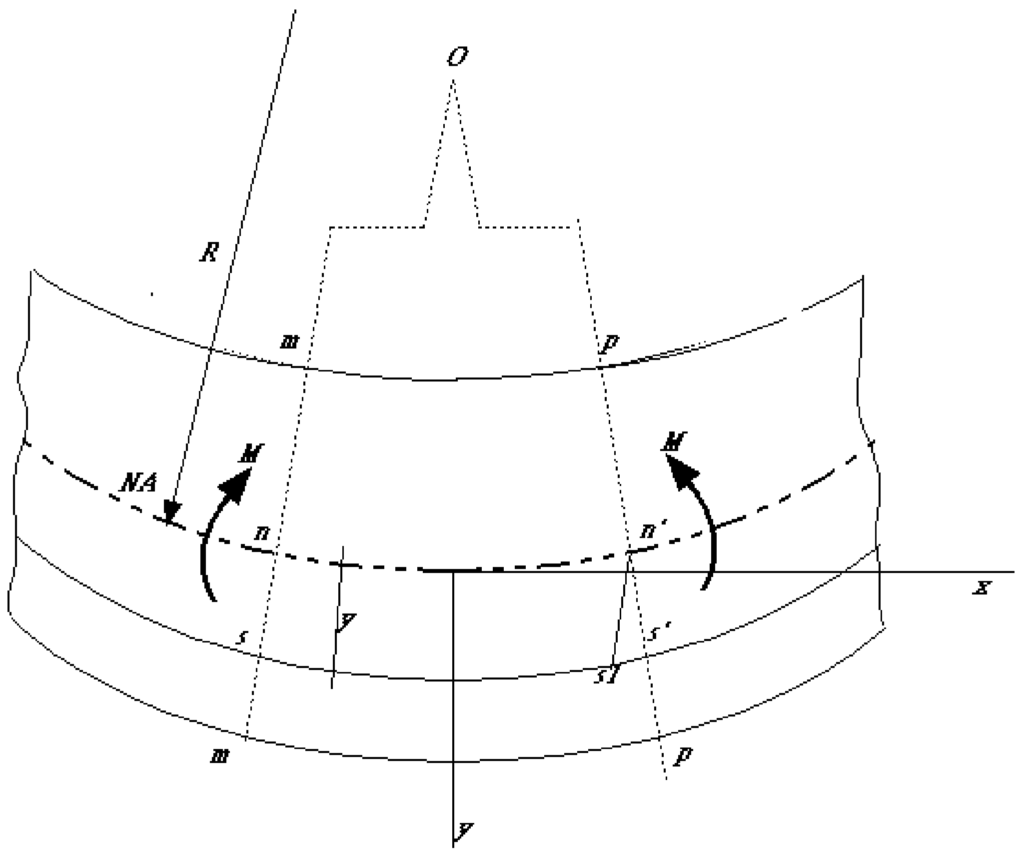
Consider the beam section in Figure 1 with lines mm and pp perpindicular to the neutral axis NA before bending.



Beam Section

Figure 1

Applying a bending moment M to the beam causes the beam to deform into a circular arc of radius R. Lines mm and nn rotate through a small angle (θ) and now pass through the origin point O, but remain straight in accordance with assumption 2, shown in Figure 2. As bending occurs, the top portion mn shortens slightly and the bottom surface mn lengthens slightly by the same amount. The neutral axis nn' remains unchanged in length.



Beam Section Bent by Moment M

Figure 2

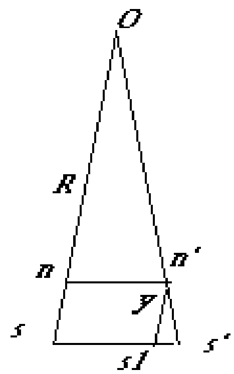
Now consider any surface s_1s' a distance y away from the neutral axis nn'. Construct the line n' s_1 parellel to line mm. Noting from assumption 1 that deflections are small we can see that triangles nOn' is similar to $s_1n's'$.

From the definition of strain, ε=ΔL/L, where ΔL = $s's_1$, and L = ss'.

$$\varepsilon = \frac{s's_1}{ss'}$$

Because triangles are similar, y and R are in the same proportion, then:

$$\varepsilon = \frac{y}{R} \qquad 1$$



Recalling Hooke's Law that stress is proportional to strain, we can easily write the stress at any point a distance y away from the neutral axis as $\sigma = E * \varepsilon_y$:

Substituting for ε gives: $$\sigma = E \cdot \frac{y}{R} \qquad 2$$

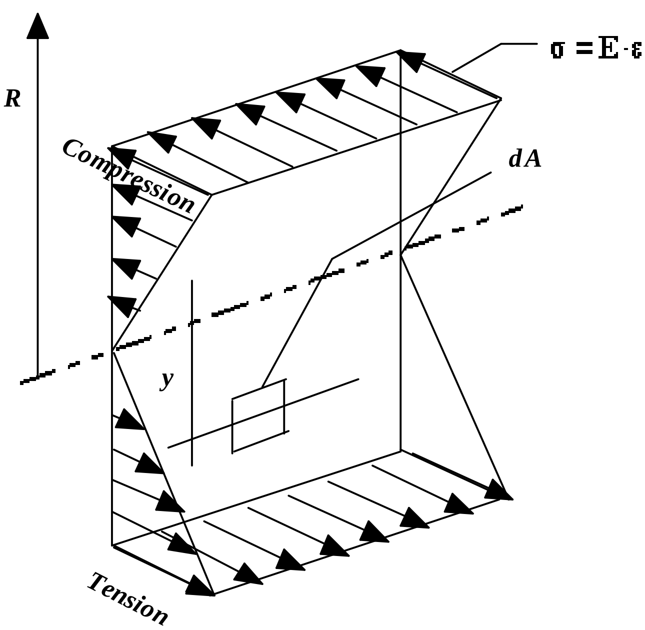


Figure 3

Let dA denote an elemental area on the cross section a distance y from the neutral axis. The force acting on this area is simply the stress at y times the area dA.

$$F_{dA} = \sigma \cdot dA$$

Substituting for stress gives:

$$F_{dA} = E \cdot \frac{y}{R} \cdot dA$$

Summing all moments accross the cross section and setting them equal the external moment M gives:

ie. Force = F_{dA}, distance = y, M = FdA * y

$$\left(\int \frac{E}{R} \cdot y \, dA\right) \cdot y = M$$

$$\left(\int \frac{E}{R} \cdot y^2 \, dA\right) = M \qquad 3$$

By definition, $\int y^2 \, dA$ is the moment of inertia I

Thus

$$M = \frac{E}{R} \cdot I \qquad 4$$

Rewriting gives:

$$\frac{1}{R} = \frac{M}{E \cdot I} \quad \text{or} \quad E \cdot \frac{I}{M} = R \qquad 5$$

where 1/R is the curvature of the beam.

Recalling that

$$\sigma = E \cdot \frac{y}{R}$$

Solving for R gives:

$$E \cdot \frac{y}{\sigma} = R \qquad 6$$

Thus we can eliminate R and write:

$$E \cdot \frac{y}{\sigma} = E \cdot \frac{I}{M}$$

Solving for stress gives:

$$\sigma = y \cdot \frac{M}{I} \qquad 7$$

Setting y = height/2 = c :for the maximum distance away from the neutral axis gives:

$$\sigma = \frac{M \cdot c}{I} \qquad \textbf{QED} \qquad 8$$

We can now define the section modulus S as I/c, which finally gives the bending stress as:

$$\sigma = \frac{M}{S} \qquad 9$$

Relationship between bending and shear

$\frac{dM}{dx} = V$ Shear force is the rate of change of bending a

$\frac{dV}{dx} = -q$ Load intensity with a negative sign, is the rate of change of shear force b

Derive the differential equation for deflection

Given:

Let the curve AmB in Figure 4 represent the neutral axis of a deflected beam after bending.

Assumptions :

Curvature at any point depends only on the magnitude of the bending moment M at that point.

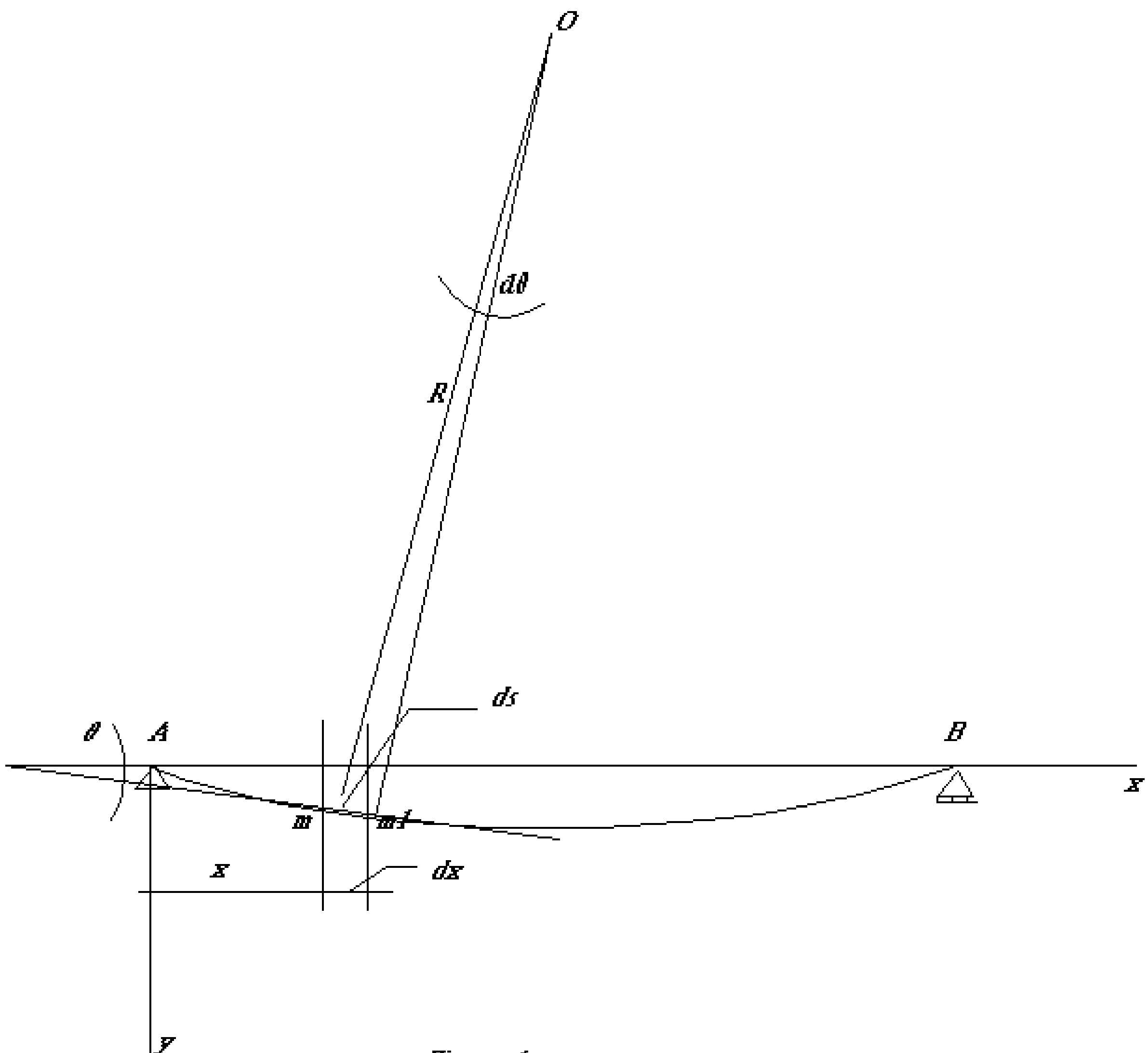


Figure 4

From equation 5

$$\frac{1}{R} = \frac{M}{E \cdot I} \qquad 5$$

Consider two adjacent points m and m1 a distance ds apart on the deflection curve. Let the angle between the tangent to the deflection curve at m and the x axis be denoted as θ. Then the angle between the two normals m and m1 must be dθ. The intersection of these normals at point O defines the center of curvature and defines the length R being the radius of curvature Then:

$$ds = R \cdot d\theta$$

The sign convention is taken such that a positive increment is ds corresponds to a decrease in dθ, thus

$$\frac{1}{R} = -\frac{d\theta}{ds} \qquad 10$$

Recalling the basic assumptions for bending that deflections are very small indicates that the deflection curve is relatively flat and angles are small. In this case it is sufficiently accurate for

$$ds = dx \qquad \theta = \tan(\theta) = \frac{dy}{dx}$$

Substituting into equation 10 gives:

$$\frac{1}{R} = -\frac{d\theta}{ds} = -\frac{d\left(\frac{dy}{dx}\right)}{dx} = -\frac{d^2y}{dx^2}$$

And equation 5 becomes:

$$\frac{M}{EI} = -\frac{d^2y}{dx^2}$$

Rewriting is standard from gives

$$EI \cdot \frac{d^2y}{dx^2} = -M \qquad 11$$

This is the differential equation of the deflection curve and must be intergrated for each specific moment case to find the beam deflection of a beam.

Differentiating equation 11 twice with respect to x and combining with equations a and b gives:

$$EI \cdot \frac{d^3y}{dx^3} = -V \qquad 12$$

$$EI \cdot \frac{d^4y}{dx^4} = q \qquad 13$$

For the case of a uniformly loaded beam, determine the deflection equation along the x axis from 0 to L.

From the free body diagram below we can write the moment at any point x along the length.

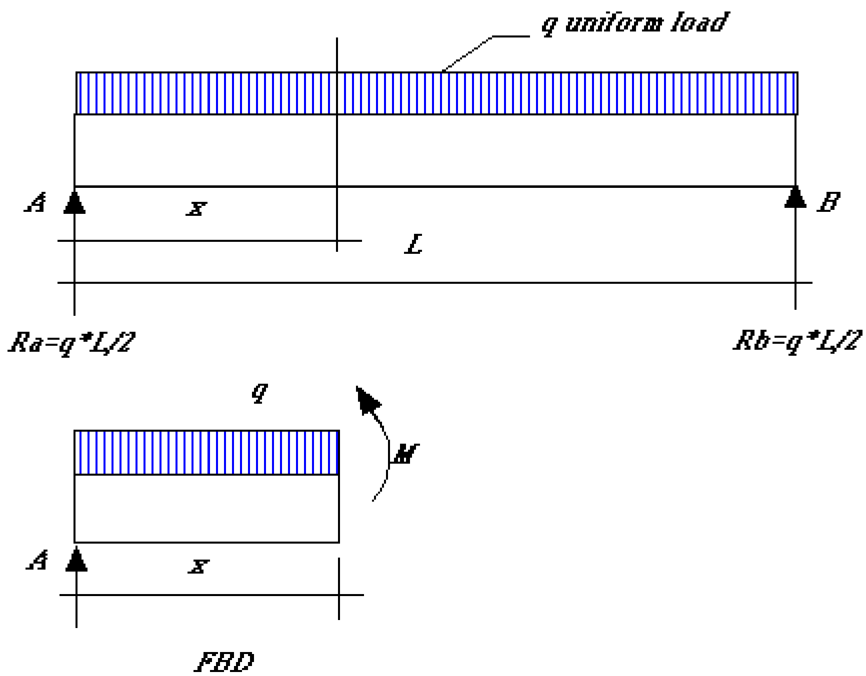


Figure 5

Summing moments at x with counterclockwise moment as positive we have:

$$R_a \cdot x - q \cdot x \cdot \frac{x}{2} = M$$

Then

$$M = \frac{q \cdot L \cdot x}{2} - \frac{q \cdot x^2}{2}$$

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Substituting for M in equation 11 gives

$$EI \cdot \frac{d^2 y}{dx^2} = \frac{-q \cdot L \cdot x}{2} + \frac{q \cdot x^2}{2}$$

Intergrating once gives the slope of the beam θ at any point x along the length.

$$\int EI\,dx\cdot\frac{d^2y}{dx^2} = \int \frac{-q\cdot L\cdot x}{2}\,dx + \int \frac{q\cdot x^2}{2}\,dx$$

$$\theta = EI\cdot\frac{dy}{dx} = \frac{-1}{4}\cdot q\cdot L\cdot x^2 + \frac{1}{6}\cdot q\cdot x^3 + C \qquad 15$$

From the boundary conditions shown in Figure 4, it is clear that the slope θ is 0 at x = L/2, or

$$EI\cdot\frac{dy}{dx} = 0 = \frac{-1}{4}\cdot q\cdot L\cdot\left(\frac{L}{2}\right)^2 + \frac{1}{6}\cdot q\cdot\left(\frac{L}{2}\right)^3 + C$$

$$\frac{-1}{4}\cdot q\cdot L\cdot\left(\frac{L}{2}\right)^2 + \frac{1}{6}\cdot q\cdot\left(\frac{L}{2}\right)^3 + C = 0$$

$$C = \frac{1}{24}\cdot q\cdot L^3$$

Substituting into 15 and intergrating again:

$$y\cdot EI = \int \frac{-1}{4}\cdot q\cdot L\cdot x^2 + \frac{1}{6}\cdot q\cdot x^3 + \frac{1}{24}\cdot q\cdot L^3\,dx$$

$$y\cdot EI = \left[\frac{1}{24}\cdot q\cdot x\cdot\left(-2\cdot L\cdot x^2 + x^3 + L^3\right)\right] + C_1$$

Again from Figure 4 we see that deflection y = 0 at x = 0. Subsittuting gives

$$0 = \frac{-1}{12}\cdot q\cdot L\cdot 0^3 + \frac{1}{24}\cdot q\cdot 0^4 + \frac{1}{24}\cdot q\cdot L^3\cdot 0 + C_1$$

$$C_1 = 0$$

The maximum deflection occurs at the center - L/2. Substituting gives:

$$y\cdot EI = \left[\frac{1}{24}\cdot q\cdot\frac{L}{2}\cdot\left[-2\cdot L\cdot\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^3 + L^3\right]\right]$$

$$y\cdot EI = \frac{5}{384}\cdot q\cdot L^4$$

$$y = \frac{5\cdot q\cdot L^4}{384\cdot EI}$$ ***QED***